J. Phys.: Condens. Matter **11** (1999) 6815–6821. Printed in the UK PII: S0953-8984(99)02218-3

Critical currents and voltages in weakly nonlinear inhomogeneous media

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Received 7 March 1999, in final form 28 May 1999

Abstract. A random mixture of two components is considered. It is assumed that both these components have current–voltage characteristics which contain weak nonlinear terms of a power-law type. General results for the effective nonlinear susceptibility as well as for critical current and voltage, defined as the crossovers from linear to nonlinear behaviour are obtained, both above and below the percolation threshold. They agree with the results obtained previously for some less general composites. New results for the mixture of 'nonlinear insulator'+'linear metal' are found. All these results are valid in the low-field limit. For larger fields it is shown that the exponent x describing the scaling of critical current as a function of conductance obeys the relation: $x \leqslant (d-1)\nu/t$ for a random metal–insulator composite and $x \geqslant 1-\nu/q$ for a superconductor–normal conductor composite (d is dimensionality, ν is the percolation correlation length exponent and t and q are conductivity critical exponents for metal–insulator and superconductor–normal conductor percolation, respectively).

In recent years there has been an increasing interest in nonlinear inhomogeneous media. Below we consider the case of the so-called weak nonlinearity. The simplest case is cubic nonlinearity, defined by a current density/field relationship of the form

$$\vec{j}(\vec{r}) = \sigma(\vec{r})\vec{K}(\vec{r}) + \chi(\vec{r})|\vec{K}(\vec{r})|^2\vec{K}(\vec{r})$$

relating the current density $\vec{j}(\vec{r})$ and electric field $\vec{K}(\vec{r})$ at any point $\vec{r}$ of the medium, where $\sigma(\vec{r})$ is the linear component of the local $\vec{j}$ versus $\vec{K}$ characteristic and $\chi(\vec{r})$ is called the local nonlinear susceptibility. The condition 'weak' requires $\sigma(\vec{r}) \gg \chi(\vec{r})|\vec{K}(\vec{r})|^2$. Stroud and Hui [1] and Aharony [2] have shown that in the limit of low fields the overall $\vec{j}$ versus $\vec{K}$ dependence is also cubic, $\vec{j} = \sigma\vec{K} + \chi|\vec{K}|^2\vec{K}$. The effective conductivity σ is given by $\sigma = \langle\sigma(\vec{r})\vec{K}(\vec{r})^2\rangle/\langle\vec{K}(\vec{r})\rangle^2$, whereas the effective nonlinear susceptibility, χ, is related to the fourth moment of the local field distribution,

$$\chi = \langle\chi(\vec{r})\vec{K}(\vec{r})^4\rangle/\langle\vec{K}(\vec{r})^4\rangle.$$

An example of such a nonlinear medium is a two-component mixture of well conducting ('metallic') and poorly conducting ('insulating') components, which both display weak cubic nonlinearity: $\vec{j} = \sigma_m\vec{K} + \chi_m|\vec{K}|^2\vec{K}$ for the metallic component and $\vec{j} = \sigma_d\vec{K} + \chi_d|\vec{K}|^2\vec{K}$ for the insulating (dielectric) component, where $\sigma_m \gg \sigma_d$, and χ_m and χ_d denote nonlinear susceptibilities of the components. One of the approaches frequently used to study a random system is to map it onto a random resistor network. The network usually takes the form of a d-dimensional lattice in which the bonds of length a_0 are occupied in a random way by either metallic or insulating components. In the network notation the effective susceptibility can be rewritten as

$$\chi = \frac{L^{4-d}}{V^4}\left(\chi_m \sum_{i\in M} V_i^4 + \chi_d \sum_{i\in D} V_i^4\right) = L^{4-d}(\chi_m W_{M2} + \chi_d W_{D2})$$

where V_i is the voltage which appears on bond i when the network is biased by the external voltage V and L is the lattice size (the number of bonds along one of network edges). W_{M2} and W_{D2} are the second moments of voltage distributions. In general the kth-order moments of these distributions are defined as [3, 4]

$$W_{Dk} = \sum_{i\in D}\left(\frac{V_i}{V}\right)^{2k} \qquad W_{Mk} = \sum_{i\in M}\left(\frac{V_i}{V}\right)^{2k}$$

for $k = 0, 1, 2, 3, \ldots$, where the summation is restricted to the bonds which are occupied by either insulating ($i \in D$) or metallic ($i \in M$) component. In the limit of low fields it was shown that the moments W_{Dk}, W_{Mk}, which are defined for the nonlinear problem, can be replaced by such moments found for the corresponding linear problem [1, 2] (i.e. for $\chi_d = \chi_m = 0$). In the linear problem the behaviour of the moments W_{Dk}, W_{Mk} was found both above and below the percolation threshold [3–10]. It is summarized in table 1.

Table 1. Scaling of moments of voltage distributions near the percolation threshold.

	$p < p_c$	$p > p_c$
W_{Dk}	$L^{d-2k}\lvert\tau\rvert^{q(2k)-2kq}$	$L^{d-2k}\tau^{q(2k)-2kq}$
W_{Mk}	$L^{d-2k}h^{2k}\lvert\tau\rvert^{t(2k)-2k(t+q)}$	$L^{d-2k}\tau^{t(2k)}$

In this table p is the volume fraction of the metallic component, p_c is the percolation threshold, $\tau \equiv p - p_c$, and $h \equiv \sigma_d/\sigma_m$ is a small parameter. The exponents $t(2k)$ and $q(2k)$ are related to the multifractal exponents $p(2k)$ and $\zeta(2k)$ defined by de Arcangelis *et al* [11, 12]

$$t(2k) \equiv (d-2k)\nu + p(2k) \tag{1}$$

$$q(2k) - 2kq \equiv (d-2k)\nu + \zeta(2k)\nu \tag{2}$$

where ν is the percolation correlation length exponent. Important special cases are $t(2) \equiv t$ and $q(2) \equiv q$, which characterize the linear conductivity behaviour above and below the percolation threshold, [$\sigma \sim \sigma_m\tau^t$ for $p > p_c$ and $\sigma \sim \sigma_d|\tau|^{-q}$ for $p < p_c$ [13, 14], and the resistance noise exponents $\kappa = 2t - t(4)$, and $\kappa' = 2q - q(4)$ [15, 16]. The relations in table 1 enable one to find the behaviour of the effective cubic-nonlinear susceptibility in the neighbourhood of the percolation threshold [10, 17]

$$\chi \sim \chi_m\tau^{t(4)} + \chi_d\tau^{q(4)-4q} = \chi_m\tau^{2t-\kappa} + \chi_d\tau^{-2q-\kappa'} \qquad \text{for } p > p_c \tag{3}$$

$$\chi \sim \chi_m h^4|\tau|^{t(4)-4(t+q)} + \chi_d|\tau|^{q(4)-4q} = \chi_d|\tau|^{-2q-\kappa'} + \chi_m h^4|\tau|^{-\kappa-2t-4q} \qquad \text{for } p < p_c. \tag{4}$$

When studying experimental nonlinear I versus V characteristics one usually defines a critical field K_c which is the value of K at which the nonlinear contribution becomes a part ε of the linear contribution [18], i.e. $\chi K_c^3 = \varepsilon\sigma K_c$. The critical field $K_c \sim (\sigma/\chi)^{1/2}$ is related to the critical voltage $K_c = V_c/(a_0 L)$ and to the critical current $I_c = V_c G$. For example, for a mixture of nonlinear metal ($\sigma_m > 0$, $\chi_m > 0$) and ideal insulator ($\sigma_d = 0$, $\chi_d = 0$) we obtain

$$I_c \sim G[(\sigma_m\tau^t)/(\chi_m\tau^{2t-\kappa})]^{1/2} \sim G\tau^{(\kappa-t)/2} \sim G^{(\kappa/t+1)/2}$$

in agreement with the results of Aharony [2] and Blumenfeld and Bergman [19]. Another well studied example is the mixture of a perfect conductor and a nonlinear normal conductor

[20, 21] It can be obtained from our two-component system assuming $\sigma_d > 0$, $\chi_d > 0$, and $\sigma_m = \infty$, $\chi_m = 0$. Nonlinear behaviour is observed below the percolation threshold. For $p < p_c$ the critical field scales as

$$K_c \sim [(\sigma_d|\tau|^{-q})/(\chi_d|\tau^{-2q-\kappa'}|)]^{1/2} \sim |\tau|^{(q+\kappa')/2}$$

in agreement with the result of Hui [20]. Another studied example was the composition of 'nonlinear metal' and 'linear insulator' [22–24]. One may imagine also the relevant case of 'nonlinear insulator'+'linear metal'. Assuming $\sigma_d > 0$, $\chi_d > 0$, and $\sigma_m \gg \sigma_d$, $\chi_m = 0$ we obtain

$$V_c \sim [(\sigma_m \tau^t)/(\chi_d \tau^{-2q-\kappa'})]^{1/2} \sim \tau^{(2q+t+\kappa')/2} \sim G^{(2q/t+1+\kappa'/t)/2}$$

for $p > p_c$, and

$$V_c \sim [(\sigma_d|\tau|^{-q})/(\chi_d|\tau|^{-2q-\kappa'})]^{1/2} \sim |\tau|^{(q+\kappa')/2} \sim G^{-(\kappa'/q+1)/2}$$

for $p < p_c$. The values of exponents that describe critical voltage V_c as a function of conductance G are summarized in table 2.

Table 2. Values of exponents that describe scaling of critical voltage V_c as a function of conductance G for nonlinear insulator+linear metal random system. In the calculation, values of critical exponents t, q and κ' from the literature [5] were used.

d	$(\kappa'/t + 1 + 2q/t)/2$	$-(\kappa'/q + 1)/2$
2	1.94(4)	−0.94(4)
3	1.08(2)	−0.93(7)

These results can be generalized to higher-order nonlinearity. If we assume $\vec{j} = \sigma_m \vec{K} + \chi_m |\vec{K}|^{2k-2} \vec{K}$ for 'metal' and $\vec{j} = \sigma_d \vec{K} + \chi_d |\vec{K}|^{2k-2} \vec{K}$ for 'insulator', $k = 2, 3, \ldots$, then the total power Π dissipated in the network is the sum of the powers dissipated in all of its bonds

$$\Pi = (a_0 L)^d \langle j \rangle \langle K \rangle = (a_0 L)^d (\sigma \langle K \rangle^2 + \chi \langle K \rangle^{2k}) = a_0^d \sum_i (\sigma_i |K_i|^2 + \chi_i |K_i|^{2k})$$

where $\sigma_i = \sigma_m$, $\chi_i = \chi_m$ for metallic bonds and $\sigma_i = \sigma_d$, $\chi_i = \chi_d$ for insulating bonds. From the above balance the generalized formula for the effective higher-order nonlinear susceptibility can be obtained [2, 25]

$$\chi = L^{-d} \sum_i \chi_i (|K_i|/\langle K \rangle)^{2k} = (\chi_m W_{Mk} + \chi_d W_{Dk}) L^{2k-d}.$$

For $\sigma_d = 0$, $\chi_d = 0$, i.e. for the first nonlinear random resistor network that was considered [1, 2, 19], this formula can be rewritten in terms of the higher-order cumulants of conductivity fluctuations, because the moments W_{Mk} relate the cumulant $\langle \delta\sigma^k \rangle_c$ of the overall conductivity fluctuations to the cumulant $\langle \delta\sigma_m^k \rangle_c$ of the local conductivity fluctuations [15, 26]

$$\langle \delta\sigma^k \rangle_c \sim L^{(2-d)k} \langle \delta G^k \rangle_c \sim L^{(2-d)k} W_{Mk} \langle \delta\sigma_m^k \rangle_c.$$

Thus we obtain the relation

$$\chi \sim L^{d(k-1)} \langle \delta\sigma^k \rangle_c$$

which only in part agrees with that of Blumenfeld and Bergman [19]. They incorrectly proposed $\chi \sim L^d \langle \delta\sigma^k \rangle_c$, which made the effective parameter χ size dependent and has led to incorrect analyses performed in some papers on related subjects [27–29].

Now we can rewrite equations (3) and (4) in a form valid for arbitrary order nonlinearity

$$\chi \sim \chi_m \tau^{t(2k)} + \chi_d \tau^{q(2k)-2kq} \qquad \text{for } p > p_c$$
$$\chi \sim \chi_m h^{2k} |\tau|^{t(2k)-2k(t+q)} + \chi_d |\tau|^{q(2k)-2kq} \qquad \text{for } p < p_c.$$

These agree with [29] and [30] and disagree with [27] and [28]. Again, the critical field K_c can be defined as a field at which the linear and nonlinear terms become comparable, $K_c = (\varepsilon\sigma/\chi)^{1/(2k-2)}$. In the case $\sigma_d = \chi_d = 0$ (random resistor network) we obtain for $p > p_c$

$$K_c \sim \left(\frac{\sigma_m \tau^t}{\chi_m \tau^{t(2k)}}\right)^{1/(2k-2)} \sim \tau^{[t-t(2k)]/(2k-2)} \tag{5}$$

or equivalently $I_c \sim G K_c \sim G^x$, with exponent $x = 1 + [1 - t(2k)/t]/(2k-2)$, in agreement with [2]. In the case $\sigma_m = \infty$, $\chi_d = 0$ (random resistor superconductor network) we obtain for $p < p_c$

$$K_c \sim \left(\frac{\sigma_d |\tau|^{-q}}{\chi_d |\tau|^{q(2k)-2kq}}\right)^{1/(2k-2)} \sim |\tau|^{[-q-q(2k)+2kq]/(2k-2)}. \tag{6}$$

Eventually for the mixture of 'nonlinear insulator'+'linear metal' ($\sigma_d > 0$, $\chi_d > 0$, $\sigma_m \gg \sigma_d$, $\chi_m = 0$) we obtain

$$K_c \sim [(\sigma_m \tau^t)/(\chi_d \tau^{q(2k)-2kq})]^{1/(2k-2)} \sim \tau^{[t-q(2k)+2kq]/(2k-2)} \tag{7}$$
$$I_c \sim G K_c \sim G^{1+[1-q(2k)/t+2kq/t]/(2k-2)}$$

for $p > p_c$ and

$$K_c \sim [(\sigma_d |\tau|^{-q})/(\chi_d |\tau|^{q(2k)\tau-2kq})]^{1/(2k-2)} \sim |\tau|^{[-q-q(2k)+2kq]/(2k-2)} \tag{8}$$
$$I_c \sim G K_c \sim G^{1+[q(2k)/q+1-2k]/(2k-2)}$$

for $p < p_c$. Since the exponents that describe I_c versus G are related (via equations (1) and (2)) to the multifractal exponents $p(2)/\nu$ and $\zeta(2k)$, their values now can be calculated. For $k = 2$, 3 and ∞, they are summarized in table 3.

Table 3. Values of exponents that describe scaling of critical current I_c as a function of conductance G for 'linear metal'+'nonlinear insulator' mixture for various types of nonlinearity. In the calculations, the values of multifractal exponents from [5] were used: $p(2)/\nu = 1.20(3)$, $\zeta(2) = 1.89(3)$, $\zeta(4) = 1.55(3)$, $\zeta(6) = 1.42(3)$ in 3D and $p(2)/\nu = \zeta(2) = 0.98(2)$, $\zeta(4) = 0.82(2)$, $\zeta(6) = 0.77(2)$ in 2D. For $k = \infty$ we assumed $\zeta(\infty) = 1/\nu$ [11, 12, 16], and $\nu = 4/3$ in 2D or $\nu = 0.88(1)$ in 3D [31].

	$p > p_c$	$p < p_c$
k	$1 + [1 - q(2k)/t + 2kq/t]/(2k-2)$	$1 + [q(2k)/q + 1 - 2k]/(2k-2)$
2	2.08(2) (3D), 2.94(4) (2D)	0.07(7) (3D), 0.06(4) (2D)
3	1.75(2) (3D), 2.47(4) (2D)	0.01(5) (3D), 0.03(4) (2D)
∞	1.45(1) (3D), 2.02(2) (2D)	−0.12(4) (3D), −0.02(2) (2D)

All the above calculations are only valid in the limit of low fields; only in this case can the distribution of fields in nonlinear systems be replaced by the distribution of fields in the corresponding linear problem. Theoretically, the low-field limit can be always approached for sufficiently small values of parameter ε, for which the critical quantities are evaluated. In practice, reasonable values of ε are of the order of 0.1. This requires larger fields for which it may happen that the system is outside the low-field limit. This problem will now be discussed in more detail.

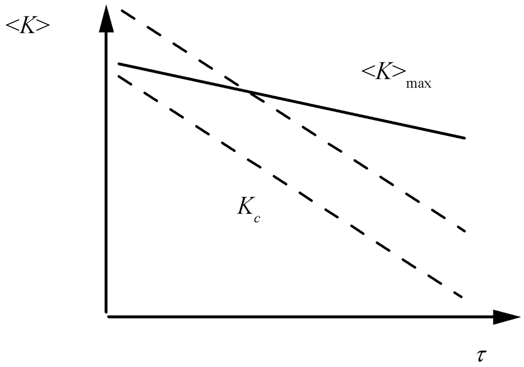

Figure 1. Log–log plots of critical fields K_c (dashed lines, equation (5)) and maximum average field $\langle K\rangle_{\max}$ (solid line, equation (9)) versus τ for a nonlinear random resistor network. Critical fields are drawn for larger (upper line) and smaller (lower line) values of parameter ε for which K_c is evaluated.

In the low-field limit we shall have $|\vec{K}(\vec{r})| \leqslant [\varepsilon_l \sigma(\vec{r})/\chi(\vec{r})]^{1/(2k-2)}$, where ε_l is a small parameter, at any point $\vec{r}$ of the system. In the case of percolation systems this condition provides an estimate of the maximum average field $\langle K\rangle_{\max}$ that may be applied to the system. This is because for a percolation system the distribution of local fields is fairly well known. For example, for a classical random resistor network ($\sigma_d = 0$) the maximum fields appear at the so-called singly connected bonds (SCBs). These bonds carry the largest currents in the system: any SCB carries the whole current from the area of size ξ^{d-1}, where ξ is the percolation correlation length. Since $\xi \sim |\tau|^{-\nu}$ we have

$$K_{\mathrm{SCB}} \sim \langle j\rangle \xi^{d-1} \sim \sigma\langle K\rangle \xi^{d-1} \sim \langle K\rangle |\tau|^{t-(d-1)\nu}.$$

This field must not exceed the maximum local field allowed in the system: $K_{\mathrm{SCB}} \leqslant K_{loc}^{\max} = [\varepsilon_l \sigma_m/\chi_m]^{1/(2k-2)}$. Consequently, the average field $\langle K\rangle$ must not exceed

$$\langle K\rangle_{\max} = K_{loc}^{\max} |\tau|^{\nu(d-1)-t}. \tag{9}$$

This value should be now compared with that of equation (5). From the theory of multifractal exponents we know that $p(2k > 2) > p(2)(2k-1)$ [11, 12, 15]. The comparison shows then that $K_c > \langle K\rangle_{\max}$ at least for $\tau \to 0$ ($p \to p_c$) (see figure 1). This means that the system enters the region where theory does not apply (somewhere inside the system the local field exceeds its maximum value $K_{loc}^{\max}$) before the critical field K_c given by equation (5) is reached: for large ε the critical field may scale in a way different from that of equation (5). However, the theory says that the critical field must not be smaller than $\langle K\rangle_{\max}$. Consequently, the exponent x in the I_c versus G relation must obey

$$x \leqslant \frac{\nu}{t}(d-1) \tag{10}$$

rather than the relation below equation (5).

In another example, the random resistor superconductor network ($\sigma_m = \infty$), the maximum fields appear at the so-called singly disconnected bonds (SDBs). These bonds are biased by the whole voltage that appears on the piece of network of size equal to the percolation correlation length ξ. We have $K_{\mathrm{SDB}} \sim \langle k\rangle \xi \sim \langle K\rangle |\tau|^{-\nu}$. This field must not exceed the maximum

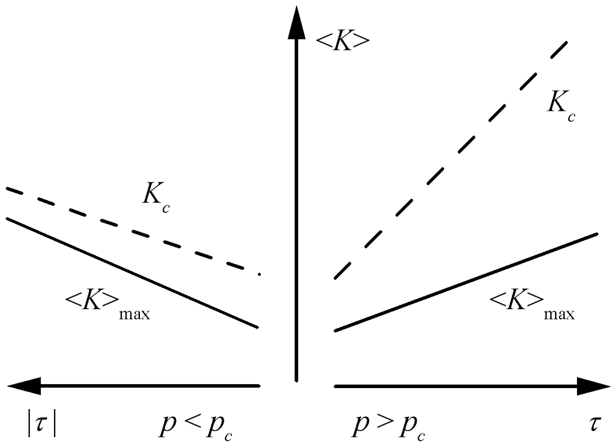

Figure 2. Log–log plots of critical field K_c (dashed lines) and maximum average field $\langle K\rangle_{\max}$ (solid lines) versus τ for the mixture of linear conductor+nonlinear insulator. The lines are drawn according to equations (7), (8) and (11).

allowed local field, which for this case is given by $K_{loc}^{\max} = [\varepsilon_l \sigma_d / \xi_d]^{1/(2k-2)}$. The maximum average field is then

$$\langle K\rangle_{\max} = K_{loc}^{\max} |\tau|^{\nu}. \tag{11}$$

Comparison with equation (6) shows that $K_c > \langle K\rangle_{\max}$ at least for $|\tau| \to 0$ ($p \to p_c$) (this conclusion is a direct consequence of the relation $\zeta(2k > 2) > \zeta(2)$, which comes from multifractals [11, 12]). If so, the scaling of critical voltage in the random conductor superconductor network may be different from that of equation (6). We can only conclude that for a large ε the exponent which describes this scaling is not larger than ν. Alternatively, the exponent x in the I_c versus G relation should obey the inequality

$$x \geqslant 1 - \frac{\nu}{q}. \tag{12}$$

Finally, let us consider the case of the network of nonlinear insulator + linear conductor. Since metal is linear, only the fields in the insulating phase need be considered: equation (11) describes the maximum average field both above and below the percolation threshold. Comparison with equations (7) and (8) leads us again to the conclusion that the system leaves the low-field limit before the critical field K_c is reached (see figure 2).

All the above examples show that, in practice, the region in which the critical quantities scale in the manner described by the theory of weakly nonlinear random media is never approached. The fields required to measure the critical quantity are usually too large to keep the system in the low-field limit, where the critical exponents given by this theory hold. For such large field the theory predicts only the relation of equation (10) for the random metal+insulator composite and of equation (12) for the superconductor+normal conductor composite.

Acknowledgment

The work was supported in part by KBN/TURz grant No U-5232/DS.

References

[1] Stroud D and Hui P M 1988 *Phys. Rev.* B **37** 8719
[2] Aharony A 1987 *Phys. Rev. Lett.* **58** 2726
[3] Kolek A 1992 *Phys. Rev.* B **45** 205
[4] Kolek A 1996 *Phys. Rev.* B **53** 14 185
[5] Tremblay R R, Albinet G and Tremblay A-M S 1991 *Phys. Rev.* B **43** 11 546
[6] Tremblay R R, Albinet G and Tremblay A-M S 1992 *Phys. Rev.* B **45** 775
[7] Tremblay A-M S, Tremblay R R, Albinet G and Fourcade B 1992 *Physica* A **183** 398
[8] Morozovsky A E and Snarskii A A 1992 *Sov. Phys.–JETP* **75** 366
[9] Albinet G, Tremblay R R and Tremblay A-M S 1993 *J. Physique* I **3** 323
[10] Snarskii A A, Morozovsky A E, Kolek A and Kusy A 1996 *Phys. Rev.* E **53** 5593
[11] de Arcangelis L, Redner S and Coniglio A 1985 *Phys. Rev.* B **31** 4725
[12] de Arcangelis L, Redner S and Coniglio A 1985 *Phys. Rev.* B **34** 4656
[13] Efros A L and Shklovskii B 1976 *Phys. Status Solidi* b **76** 475
[14] Straley J P 1976 *J. Phys. C: Solid State Phys.* **9** 783
[15] Rammal R, Tannous C and Tremblay A-M S 1985 *Phys. Rev.* A **31** 2662
[16] Wright D C, Bergman D J and Kantor Y 1986 *Phys. Rev.* B **33** 396
[17] Levy O and Bergman D J 1994 *Phys. Rev.* B **50** 3652
[18] Gefen Y, Shih W H, Laibowitz R B and Viggiano J M 1986 *Phys. Rev. Lett.* **24** 3097
[19] Blumenfeld R and Bergman D J 1991 *Phys. Rev.* B **43** 13 628
[20] Hui P M 1990 *Phys. Rev.* B **41** 1673
[21] Hui P M 1994 *Phys. Rev.* B **49** 15 344
[22] Yang C S and Hui P M 1991 *Phys. Rev.* B **44** 12 559
[23] Yu K W and Hui P M 1994 *Phys. Rev.* B **50** 13 327
[24] Chung K H and Hui P M 1994 *J. Phys. Soc. Japan* **63** 2002
[25] Bergman D J 1989 *Physica* A **157** 72
[26] Blumenfeld R, Meir Y, Aharony A and Harris A B 1987 *Phys. Rev.* B **35** 3534
[27] Zhang G M 1996 *J. Phys.: Condens. Matter* **8** 6933
[28] Zhang G M 1996 *Phys. Rev.* B **53** 20
[29] Gao L, Lu W-G and Li Z-Y 1998 *Phys. Rev.* B **59** 668
[30] Snarskii A A and Buda S I 1997 *Physica* A **241** 350
[31] Stauffer D and Aharony A 1995 *Introduction to Percolation Theory* (London: Taylor and Francis)