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Fractal-to-Euclidean Crossover in Quantum Percolation

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Extensive numerical studies of quantum percolation in 2D show no indications of localization–delocalization transition. At the percolation threshold, i.e. for $p = p_c$, the scaling curve $\beta \equiv \partial \ln g/\partial \ln L$ exhibits a fractal-like behavior. For $\ln g \ll 0$ it senses superlocalization: it has the slope $d_\phi \cong 1.14$. For $\ln g \gg 0$ it saturates at $-t/\nu + d - 2 \cong -1$, where t and ν are percolation critical exponents. For small size L ($\sim$10) of percolation cluster the distribution of variable $\lambda_1 = \operatorname{arccosh}(1/\sqrt{T_1})$, where T_1 is the first transmission eigenvalue, has the exponential tail $P(\lambda_1) \sim \exp(-\lambda_1)$, which is characteristic for chaotic cavities with one-moded leads. For intermediate sizes $P(\lambda_1)$ changes to Wigner surmise typical for metallic states. For large sizes the shape of $P(\lambda_1)$ results from the "convolution" of the first Lyapunov exponent γ_1 (which is Gaussian) and chemical length l (which has a tail for large l). For $p > p_c$ we observe a crossover from fractal-like behavior for $L \ll \xi_p$, ξ_p is the percolation correlation length, to Euclidean-like behavior, characteristic for homogeneous disorder, for $L \gg \xi_p$.

Recent series expansion studies (see e.g. [1]) suggest that there is a localization–delocalization transition in 2D quantum percolation, which is in contradiction to some numerical results (see e.g. [2]). Apart from this controversy another question arises: On one hand, quantum percolation can be considered as a kind of random potential and its conductance g is expected to follow the universal scaling curve $\beta \equiv \partial \ln g/\partial \ln L$ with the limits $\beta_E \propto d - 2 = 0$ for $\ln g \gg 0$ and $\beta_E \propto \ln g$ for $\ln g \ll 0$. On the other hand, at the threshold concentration of sites $p = p_c$ the percolation cluster (p.c.) has a fractal geometry for which the β-curve is quite different [3], $\beta_F \propto -t/\nu + d - 2 \cong -1$ for $\ln g \gg 0$ and $\beta_F \propto d_\phi \ln g$ for $\ln g \ll 0$. Here either $d_\phi = 1$ [3] or $d_\phi = \zeta_l \cong 1.15$ [4] and t, ν and ζ_l are percolation critical exponents for conductivity, correlation length, and chemical distance, respectively. In this paper, the results of extensive numerical simulation of quantum percolation on a square lattice are presented. Tight binding Hamiltonian with hopping restricted to nearest neighbors and on-site disorder are used. The conductance $\exp(\langle \ln g \rangle)$ averaged in the ensemble of 50000 configurations of $L \times L$ square lattice inserted into infinite, disorder-free L-wide strip is calculated for increasing size up to $L = 100$. The results presented in Fig. 1 reveal the following behavior: (i) For $p = p_c$ the conductance shows superlocalization [4, 5]. The exponent $d_\phi \cong 1.14$ is found in good agreement with the conjecture $d_\phi = \zeta_l$. This result is not so obvious if we note that our calculations are performed in the middle of the band, i.e. for $E = 0.5$, where by the theory only the relation $1 \leq d_\phi \leq \zeta_l$ holds [4]. (ii) For $p > p_c$ and in the limit $L \to \infty$ data follow β_E – the scaling function for 2D (Euclidean) space. This means that no indications of localization–delocalization transition is found. (iii) The limit $L \to \infty$ in (ii) reads $L \gg \xi_p$, where ξ_p is the percolation correlation length. Since the latter diverges at the percolation threshold, $\xi_p \sim (p - p_c)^{-\nu}$, we observe that for $p \gg p_c$ (e.g.

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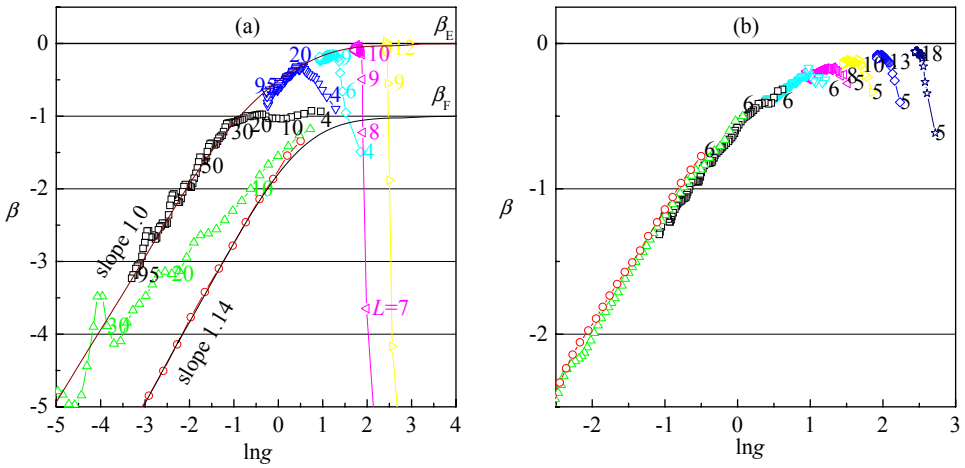

Fig. 1 (online colour). Comparison of β curves for a) quantum percolation and b) Anderson model with a box distribution of site energies (both in 2D). Symbols refer to different a) concentrations p of sites: $p = p_c = 0.5973$ (○), 0.65 (△), 0.7 (□), 0.75 (▽), 0.8 (◇), 0.85 (◁), 0.9 (▷), b) widths W of box distribution: $W = 7$ (○), 6 (△), 5 (□), 4 (▽), 3.5 (◁), 3 (▷), 2.5 (◇), 2 (☆). Labels along the traces indicate size L of the lattice. Lines drawn in a) show the expected shapes of β in Euclidean (index E) and fractal (index F) geometries

for $p = 0.9$) data "enter" β_E-curve for relatively small L of several lattice spacings. In contrast, for $p \to p_c$ lattices as large as 30×30 (see e.g. $p = 0.65$ in Fig. 1a) are too small to reach β_E-curve. (iv) In the limit $L \ll \xi_p$ data follow β_F – the "fractal" scaling curve. Unfortunately, this limit can be achieved only for $p \cong p_c$, where ξ_p is sufficiently large. This is the case described in (i). (v) What we mostly observe is the region $L \cong \xi_p$, in which there is the crossover from fractal to Euclidean scaling. Since ξ_p is a function of p the crossover curve is unique for each p. For large values of p ($p \geq 0.75$) the crossover is in the "metallic" regime, where $\beta_F \cong -1$ and $\beta_E = 0$. As a result the crossover is nonmonotonic: β rises from -1 to 0 and then drops down following β_E curve. Tuning p we may find such crossover curve which takes the shape of horizontal line. In this case ($p = 0.7$) a power law localization is observed for quite a large range of L.

The criticism that one could make against the above picture is that traces for different values of p diverge at small sizes due to entering ballistic regime and are observed also for the ordinary Anderson model with a box distribution of site energies (see Fig. 1b). When disorder (W or $1 - p$) increases the mean free path decreases and dispersion of the curves should occur for smaller and smaller sizes. Indeed, such a behavior is observed in case of box distribution in Fig. 1b, and for quantum percolation for $p = 0.9$, 0.85, and 0.8, where with $1 - p$ increasing, the size necessary to enter β_E curve drops from 12 to 9. Starting from $p = 0.75$ this tendency reverts: for $p = 0.65$ we need as much as $L = 30$ to reach the universal β_E curve. This behavior cannot be explained in terms of crossover to ballistic regime but it is consistent with the fractal-to-Euclidean crossover as we have already explained in (iii).

Another argument we would like to call in support of the picture of quantum percolation presented above is the distribution of the first Lyapunov exponent γ_1. We have chosen this quantity because for infinite percolation cluster it is the only Lyapunov

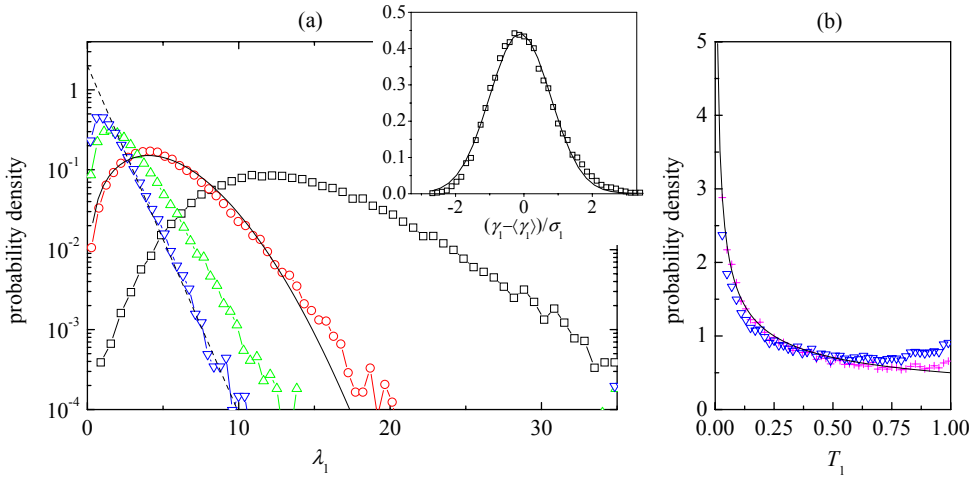

Fig. 2 (online colour). Distributions of a) variable $\lambda_1 = \operatorname{arccosh}(1/\sqrt{T_1})$ for percolation cluster at $p = p_c$ for various size $L = 75$ (□), 30 (○), 15 (△), 10 (▽), b) eigenvalue T_1 for $L = 10$ for the whole 50000 population of configurations (▽), 39780 configurations which contain SCB's (+). Lines in a) are the plots of Wigner surmise: $P(\lambda_1) = \pi\lambda_1(2\langle\lambda_1\rangle^2)\exp(-\pi\lambda_1/\langle\lambda_1\rangle)^2/4)$ with $\langle\lambda_1\rangle = 5.042$ for $L = 30$ (solid) and exponential decay $P(\lambda_1) \sim \exp(-\lambda_1)$ (dashed). Line in b) is the plot of $w(T_1) = 1/(2\sqrt{T_1})$. Inset: Gaussian fit to the distribution of $\gamma_1 = \lambda_1/l$ for $L = 75$

exponent which survives. It is well known that p.c. at criticality exhibits a one dimensional geometry. It is composed of irregular blobs connected by the so-called singly connected bonds (SCB) [6]. The latter, if cut, disjoin the cluster into separate pieces making transport impossible. For quantum transport SCB plays the role of one-mode constrictions with only one eigenchannel T_1 being open. The crossover from fractal to Euclidean behavior should be accompanied with simultaneous change of the distribution of γ_1. In strongly localized regime and for fractal geometries it should be Gaussian provided the chemical length l is involved in its definition [7, 8]. Namely, $\gamma_1 \equiv \lambda_1/l$, where $T_1 = \cosh^{-2}\lambda_1$. On the contrary for homogeneous disorder the distribution is Gaussian if it is calculated in ordinary "air distance" way, $\gamma_1 \equiv \lambda_1/L$ [9]. One can easily check this change of distribution performing simulations for e.g. $p = 0.65$ and for energies near the band edge, where the wave functions are strongly localized [10] (see also the inset of Fig. 2a). Of more interest and controversial is the crossover in weakly localized or metallic regime. In Euclidean geometries the distribution of γ_1 in this regime is Wigner surmise (WS) [9, 11]. In 2D one can approach this distribution for low disorder as the limit for decreasing L. For fractal geometry of p.c. this is not the case! Here with L decreasing WS is only an intermediate case observed for $L \cong 30$ (see Fig. 2a). As L is further decreased, a long tail which is well fitted by $P(\lambda_1) \sim \exp(-\lambda_1)$ grows for large values of λ_1. We attribute this behavior to the (fractal) structure of p.c. For sizes as small as $L \sim 10$ most configurations have only one blob connected to the rest of the systems via two SCB's. It works like a cavity with two one-moded leads. Because of rough boundaries electron diffusion in such cavity is chaotic. In this case the distribution of (the only) transmission eigenvalue is known to be $w(T_1) = 1/(2\sqrt{T_1})$ [12]. In Fig. 2b the distribution of T_1 for $p = p_c$ and $L = 10$ is shown. When (few) configurations which contain no SCB's are excluded from 50000 population the agreement with $w(T_1)$

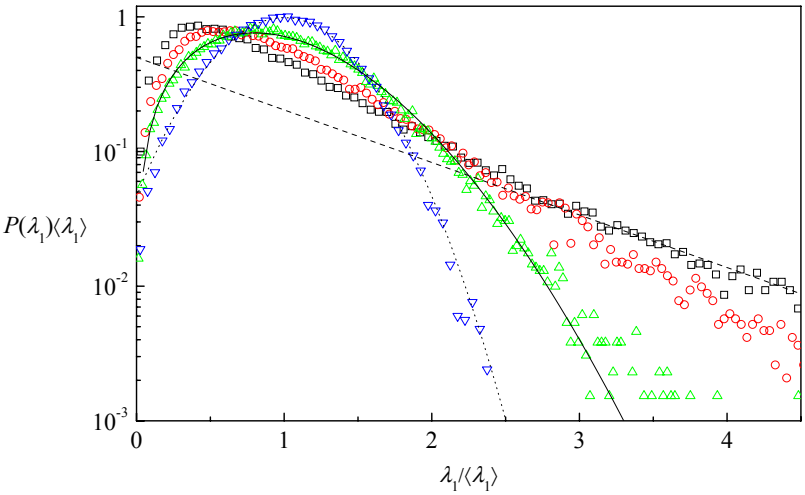

Fig. 3 (online colour). Distribution of normalized variable $\lambda_1/\langle\lambda_1\rangle$ for $p = 0.7$ and various sizes $L = 10$ (□), 20 (○), 50 (△), 160 (▽) of p.c. Lines are the plots of WS (solid), Gauss (dotted) and exponential decay $P(\lambda_1) \sim \exp(-\lambda_1)$ (dashed)

is excellent. This explains the origin of exponential tail in the distribution of λ_1: In the limit $T_1 \to 0$ we have $T_1 \cong 4\exp(-2\lambda_1)$ and the form of $P(\lambda_1) \sim \exp(-\lambda_1)$ stems directly form $w(T_1)\,\mathrm{d}T_1 = P(\lambda_1)\,\mathrm{d}\lambda_1$. We conclude that exponential tail in the distribution of λ_1 is the sign of the fractal geometry of p.c. in the metallic regime. We should expect such tails if the fractal-to-Euclidean crossover takes place in this region. A certain example is the case $p = 0.7$ for which the crossover in Fig. 1a is observed for $g \cong 1$. In Fig. 3 the distributions of λ_1 for this case are shown for various sizes. For $L = 50$ it is WS. As the size increases the distributions change towards Gaussian. These two are consistent with Euclidean behavior as data in Fig. 1a for $L > 30$ lie on β_{E}-curve. For $L < 30$ data in Fig. 1a crossover to fractal geometry. The signs of this are visible also in $P(\lambda_1)$: For small size a tail, which eventually achieves exponential limit, grows up.

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