ARTICLES

1/*f* noise in percolation and percolationlike systems

A. A. Snarskii* and A. E. Morozovsky
Department of General and Theoretical Physics, Kiev Polytechnic Institute, Prospekt Peremoga 37, 252056 Kiev, Ukraine

A. Kolek and A. Kusy
Department of Electronic Fundamentals, Rzeszów University of Technology, Wincentego Pola 2, 35-959 Rzeszów, Poland

(Received 19 October 1995)

The behavior of $1/f$ noise effective intensity in two-phase percolation systems and percolationlike systems with an exponentially wide distribution of bond resistances is reviewed. Monte Carlo simulations on random resistor networks are performed. For a two-phase system the numerical values of noise critical exponents $\kappa=1.54\pm0.025$, $\kappa'=0.61\pm0.02$, $w=6.31\pm0.25$, and $w'=6.9\pm0.25$ are found in agreement with theoretical analysis performed with the help of a hierarchical model of a two-phase percolation system. For a system with an exponentially wide spectrum of bond resistances, i.e., a system in which bonds take on resistances $r=r_0\exp(-\lambda x)$, where $\lambda\gg1$ and x is a random variable, it is assumed that in the individual resistors the noise generating mechanism obeys the form $\{\delta r^2\}\sim r^{2+\theta}$. In this case the effective noise intensity $C_e\equiv S\Omega$, where S is the relative power spectral density of system resistance fluctuations and Ω is the system volume, is given by $C_e\sim\lambda^m\exp(-\lambda\theta x_c)$, where $1-x_c$ is the percolation threshold. The exponent m is ''double universal,'' i.e., it is independent of lattice geometry and of the microscopic noise generating mechanism. Numerical simulations performed for $\theta=1$ and 0 give approximately $m\cong2.3$ and confirm this ''double universality'' of the exponent m. The connections between $1/f$ noise effective intensity and effective susceptibility in a two-phase weakly nonlinear percolation system are also established. [S1063-651X(96)04405-4]

PACS number(s): 64.60.Ak, 64.60.Ht

I. INTRODUCTION

When electrical current flows through a conducting sample the spectral density of voltage fluctuations increases in comparison to that observed in equilibrium conditions (Nyquist noise). This additional noise indicates a strange property—its relative power spectral density (RPSD) is roughly inversely proportional to frequency. For this reason this type of noise is often called $1/f$ noise. Other names are resistance (conductance) noise, current noise, excess noise, or flicker noise. The paper deals with the problem of estimating the $1/f$ noise intensity in macroscopically disordered media.

Let us recall fundamental definitions. Consider a resistor characterized by resistance R, its fluctuation δR, and the volume Ω which the resistor occupies. Then power spectral density of δR is $S_R=\{\delta R\,\delta R\}$, whereas RPSD is $S\equiv S_R/R^2$. $\{\}$ denotes the Fourier transform of the time correlation function. The *effective noise intensity* is defined as $C_e\equiv\Omega S$. With $\Omega\to0$ C_e becomes the *local noise intensity* at the point $\vec{r}$,

$$C(\vec{r})=\lim_{\Omega\to0}\Omega S. \tag{1}$$

It is well known that fluctuations δR are spatially uncorrelated [1]. Utilizing this property one can calculate the effective noise intensity in terms of the second moment of the Joule power dissipated in the medium [2],

$$C_e=\frac{\langle C(\vec{r})[\vec{E}(\vec{r})\cdot\vec{j}(\vec{r})]^2\rangle}{[\langle\vec{E}(\vec{r})\cdot\vec{j}(\vec{r})\rangle]^2}, \tag{2}$$

where $\vec{E}(\vec{r})$ and $\vec{j}(\vec{r})$ are the local electric field and current density, respectively, and averaging is over the whole sample. Thus to evaluate C_e it is necessary to know not only the spatial distribution of the local noise intensity $C(\vec{r})$ but also the distributions of currents and fields. The important information that can be read from Eq. (2) is that only these parts of the medium, in which large dc currents flow, produce large contributions to $1/f$ noise effective intensity.

The above considerations are valid also for percolation systems which recently have attracted much interest. For example, for percolation on a discrete lattice every bond i of the lattice is randomly occupied by a resistance r_i to which a small fluctuating term δr_i is attributed. RPSD of δr_i is consequently $s_i=\{\delta r_i^2\}/r_i^2$ whereas the local $1/f$ noise intensity $C_i=s_i a_0^d$, where a_0 is the lattice spacing, and d is the dimensionality of the lattice. Consequently Eq. (2) transforms to [3]

$$C_e=\Omega S=a_0^d L^d\,\frac{\Sigma_i I_i^4 s_i r_i^2}{(\Sigma_i I_i^2 r_i)^2}=a_0^d L^d\,\frac{\Sigma_i V_i^4 s_i/r_i^2}{(\Sigma_i V_i^2/r_i)^2}, \tag{3}$$

*Electronic address: snar@carrier.kiev.ua

where I_i (V_i) is the current (voltage) in a bond i when an external unit current (voltage) is applied to the lattice of size L (in units of a_0). Equation (3) enables us to write down simple composition laws which, in analogy to Kirchhoff's laws, make the calculation of $1/f$ noise RPSD for simple resistor circuits possible. Namely, when N resistors each of resistance r_i, and RPSD s_i are connected in series or parallel we have

$$S=\sum_{i=1}^{N}\left(\frac{r_i}{R}\right)^2 s_i \quad \text{or} \quad S=\sum_{i=1}^{N}\left(\frac{R}{r_i}\right)^2 s_i, \tag{4}$$

respectively, and R is the total resistance and S is RPSD of a given connection.

To evaluate C_e it is necessary to know not only a spatial distribution of the local noise intensity $C(\vec{r})$ but also the distribution of currents and fields and the solution of this problem is usually not an easy task. To our knowledge no general expression for a dependence of C_e on the composition of the disordered system with arbitrary type of randomness has been delivered until now. However, heterogeneous media which are close to the percolation threshold behave in many circumstances in a universal way. For example, it is well known [4,5] that effective conductivity σ_e of a binary (two-phase) composite with component conductivities σ_1 (metal) and $\sigma_2 \ll \sigma_1$ (''insulator'') behaves like the order parameter in the second order phase transition [6], and can be described by the scaling form

$$\sigma_e(\tau,h)=\sigma_1\mu^s F_1(\tau/\mu^u, h/\mu)=\sigma_1 h^s F(\tau/h^u), \tag{5}$$

where $\tau=(p-p_c)/p_c$ is a relative distance from the percolation threshold p_c, p is the concentration of a metal (conductivity σ_1), $h=\sigma_2/\sigma_1 \ll 1$, μ is the usual scaling parameter, and F_1 and F are scaling functions. Exponents s and u can be expressed in terms of other critical exponents t and q, $s=t/(t+q)$, $u=s/t$. The scaling function $F(z)$ approaches a finite limit for $z\to 0$, whereas it behaves like a power-type function at $z\to\pm\infty$. Expanding Eq. (5) at $h=0$ or at $\tau=0$ one can obtain relations for σ_e above ($p>p_c$), below ($p<p_c$), and at the percolation threshold in the so-called smearing region, which is the analog of the smearing region in the theory of the second order phase transition,

$$\sigma_e=\sigma_1\tau^t(A_0+A_1 h\tau^{-(t+q)}+\cdots)$$

$$\text{for } p>p_c \text{ and for } \tau>\Delta, \tag{6a}$$

$$\sigma_e=\sigma_2|\tau|^{-q}(B_0+B_1 h|\tau|^{-(t+q)}+\cdots),$$

$$\text{for } p<p_c \text{ and for } |\tau|>\Delta, \tag{6b}$$

$$\sigma_e=\sigma_1^{1-s}\sigma_2^s(D_0+D_1 h^{-1/(t+q)}\tau+\cdots)$$

$$\text{for } |\tau|<\Delta, \tag{6c}$$

where $\Delta=h^{1/(t+q)}$ is the width of the smearing region [4,5] and $A_0, A_1, B_0, B_1, D_0, D_1$ are constants of order unity. One has to point out that when $h\ll 1$ it is acceptable, in the first-level approximation, to bound the above expansions to A_0, B_0, and D_0 only. Thus the well known behavior of σ remains by this approximation reconstructed,

$$\sigma_e\sim\sigma_1\tau^t \quad \text{for } p>p_c, \quad \sigma_e\sim\sigma_2|\tau|^{-q} \quad \text{for } p<p_c. \tag{7}$$

As we see, the exponents t and q describe the critical behavior of the effective conductivity and for this reason are often called conductivity critical exponents.

Studying the behavior of $1/f$ noise in the vicinity of the percolation threshold we are interested in expressing $C_e(\tau,h)$ in forms similar to those just obtained for the effective conductivity [Eqs. (5) and (6)]. However, unlike the case of σ_e, in the case of C_e bounding its expansion to the first terms may not be adequate. For example, above the percolation threshold it has been shown [7–13] that the contribution of the insulating component which manifests itself through the higher order terms in the expansion of C_e may turn out to be not only a correction but rather an important, not negligible contribution. It is not so obvious since for $p>p_c$ a dc current flows in the medium mainly through its metallic parts.

Until now we have considered two-phase (TP) percolation media which are characterized by the following resistance distribution:

$$f(r)=p\,\delta(r-r_1)+(1-p)\,\delta(r-r_2), \quad r_2\gg r_1 \tag{8}$$

where r_1 and r_2 are resistances which each bond i of the lattice may take on in a random way with probabilities p and $1-p$. They are related to the component conductivities, $r_1=a_0^{2-d}/\sigma_1$, $r_2=a_0^{2-d}/\sigma_2$. There is, however, also a very interesting case of continuous resistance distributions. In this case the strong heterogeneity can be maintained if the ''worst'' resistance which may appear in the system is much greater than the ''best'' one. As an example one can consider here the case

$$r_i=r(x)=r_0 e^{-\lambda x}, \tag{9}$$

where r_i is the resistance of the ith bond and $x\in[0,1]$ is a random variable with a smooth distribution $D(x)$. Strong heterogeneity is provided by $\lambda\gg 1$; in this case the maximum resistance which appears in the lattice, $r_{\max}=r_0$, is many orders of magnitude larger than the minimum one, $r_{\min}=r_0\exp(-\lambda)$. As an important example, which can be just reduced to such a simplified problem, we recall the high temperature hopping conduction. In high temperatures we may neglect dispersion of energy levels on the localization centers [14] and $r-\epsilon$ hopping becomes r hopping. Although the case with a continuous distribution of bond resistances defined by Eq. (9) is not the classical percolation—it does not exhibit the percolation threshold at which one of the two components forms an infinite percolating cluster because the components as such do not exist—there is the well known approach [15–17] which allows one to reduce the problem with a continuous distribution of resistances to the standard percolation. Recently it was also shown [18,19] that for obtaining a not inconsistent percolationlike description of the behavior of effective quantities in such media it is necessary to assume that the system is in the smearing region. A model of a percolation system in this region has also been recently proposed [20].

In the present paper in Sec. II a hierarchical model of a TP percolation system ($h\neq 0$) is presented and, on its basis,

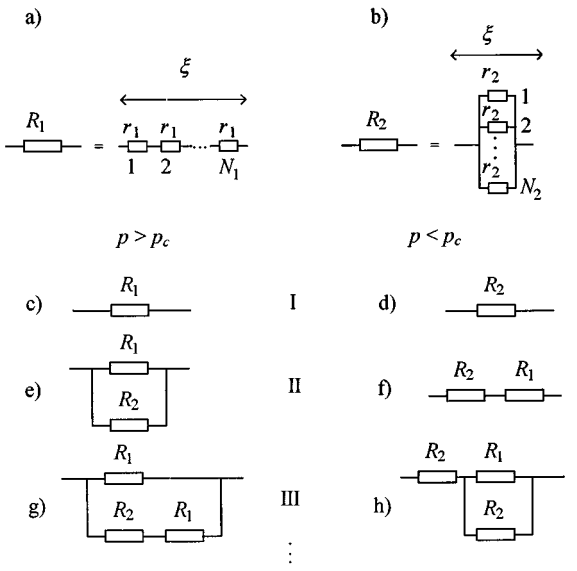


FIG. 1. Hierarchical model of a two-phase (TP) percolating system of size ξ—the percolation correlation length. (a) R_1 is the resistance of a percolating metallic bridge. It is composed of N_1 resistors r_1. (b) R_2 is the resistance of an ''insulating'' interlayer. It is composed of N_2 parallel resistances r_2. The numbers N_1 and N_2 diverge at the percolation threshold, $N_1 \sim |\tau|^{-\alpha_1}$ and $N_2 \sim |\tau|^{-\alpha_2}$. Hierarchical schemes model a TP system above (left column) or below (right column) the percolation threshold. In the first level (I) of the hierarchy a TP system is modeled by resistance R_1 for $p > p_c$ (c), or by resistance R_2 for $p < p_c$ (d). In the next levels (II,III,...) resistances R_2 and R_1 are added alternatively in parallel or in series to the elements R_1 and R_2 added in the previous step of a generation (e)–(h).

the critical behavior of the effective noise intensity C_e is evaluated. In Sec. III results of numerical simulation of a TP system are presented. In Sec. IV comments on the behavior of the nonlinear susceptibility in random weakly nonlinear two-phase systems are supplied. In Secs. V and VI the behavior of C_e in systems with an exponentially wide spectrum of bond resistances is described.

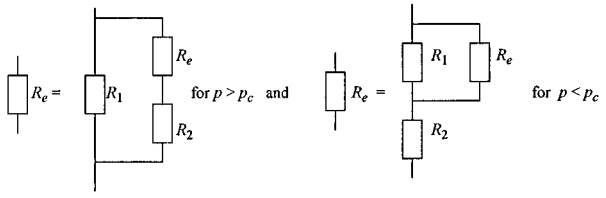
II. HIERARCHICAL MODEL OF A TWO-PHASE PERCOLATION SYSTEM

A model allowing quantitative description of current flow through a heterogeneous medium for $p > p_c$ has been developed by Skal and Shklovski [21] and de Gennes [22]. A very basic assumption of this model is that on distances of the order of percolation correlation length ξ (see Ref. [23] for a review of percolation), current flows through singly connected strings or bridges formed from a metallic component which joins nodes of the backbone of an infinite metallic cluster. Bonds contributing to the given percolating bridge are nonparallel ones and are often referred to as singly connected bonds (SCB's). When approaching the percolation threshold from above, the number N_1 of SCB's diverges as $N_1 \sim \tau^{-\alpha_1}$ and it is widely accepted that exponent $\alpha_1 = 1$ [21–24,14]. Thus that resistance R_1 of a metal bridge is $R_1 = N_1 r_1$ [see Fig. 1(a)].

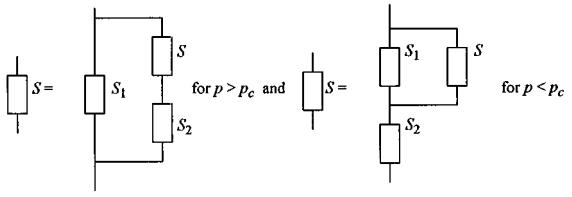
A similar model, but one which is below the percolation threshold, has also been proposed [25–27]. According to this model the main part of the voltage applied to the heterogeneous system drops on poorly conducting bonds which form an insulating interlayer placed between isolated clusters of metallic bonds. These poorly conducting bonds in a given interlayer are connected in parallel and it is accepted to call them singly disconnected bonds, since they disconnect separate metallic clusters. Thus resistance R_2 of an insulating interlayer consists of N_2 resistances r_2 connected in parallel [see Fig. 1(b)]. The number N_2 diverges at the percolation threshold, $N_2 \sim |\tau|^{-\alpha_2}$ with $\alpha_2 = 1$. It has also been proposed [28,8] to assume $\alpha_1 = \zeta_R = t - \nu(d-2)$ and $\alpha_2 = \zeta_G = q + \nu(d-2)$, where ν is the percolation correlation length exponent, $\xi \sim |\tau|^{-\nu}$ (in units of a_0). Note that in this case the models described above lead directly to Eq. (7) and this means that these simple models can give only the first main terms of the expansions of σ_e given by Eqs. (6a) and (6b).

For $h = \sigma_2/\sigma_1 = 0$ ($\sigma_2 = 0$, $p > p_c$ or $\sigma_1 = \infty$, $p < p_c$) consecutive terms of the expansion in Eq. (6) are absent. However, in real physical systems h is always finite, and further terms of the expansion may play a non-negligible role. It is even possible to have physical processes for which the critical behavior is mainly determined by the second terms of expansion in Eq. (6) [29]. A model which takes into account the second and the next terms of the expansions should include simultaneously a metal bridge and an insulating interlayer both above and below the percolation threshold [8]. The first stages of such a hierarchical (with respect to h) model of a percolation structure are also shown in Fig. 1. For $p > p_c$ current mainly flows through a metallic bridge of resistance R_1, as is seen in the first (I) level of the hierarchy [Fig. 1(c)]. In the second (II) level an insulating interlayer of resistance R_2 is added in parallel to R_1, as in Fig. 1(e). Note that voltage drops on R_1 and R_2 are in fact the same and electrical breakdown might take place faster in the insulating interlayer than in the metallic bridge [20]. It means that an element of the structure, which appears to be unimportant at first glance, giving only a small correction of the order of $h\tau^{-(t+q)} \ll 1$ to σ_e, may happen to be of essential importance. Furthermore, this argumentation is also valid even for $h = 0$, i.e., the element of the structure which does not exist from the ''σ point of view'' turns out to be an essential one. A similar situation is also possible for $p < p_c$. Here a supplementary element of the percolation structure—a metallic bridge—may become essential if a breakdown phenomenon is based upon Joule heating, i.e., in a case when a breakdown takes place once $j > j_c$, where j_c is a critical current.

The consecutive steps of the hierarchical model are presented in Fig. 1. Further steps of the hierarchy may be continued as long as necessary—each successive step represents a successive term in the expansion of σ_e given by Eq. (6). It was shown [8] that it is possible on the basis of this hierarchical model to write down a self-consistent equation, the analog of the Dyson equation [30], which includes all the members of the hierarchy. In the case of effective conductivity this equation in symbolic representation has the following form:



where R_e is related to σ_e via $\sigma_e=(a_0\xi)^{2-d}/R_e$. The solutions of these symbolic equations expanded in powers of the small parameter $h|\tau|^{-(t+q)}$ reconstruct series in Eqs. (6a) and (6b). The analog of the Dyson equation in the case of the $1/f$ noise effective intensity can also be written down [8]



where S_1 and S_2 denote $1/f$ noise RPSD's of a metal bridge and an insulating interlayer, respectively.

The calculation of $1/f$ noise RPSD for the model in Fig. 1(e) enables an estimation of C_e for $p>p_c$. In the three-dimensional (3D) case we have

$$C_e \simeq \frac{C_1 j_1^2 E_1^2 \Omega_1/\Omega + C_2 j_2^2 E_2^2 \Omega_2/\Omega}{\sigma_e^2 \langle E\rangle^4}, \tag{10}$$

where j_1 and E_1 are the electric current density and field in the bridge, while j_2 and E_2 are those in the interlayer, and $\Omega_1=a_0^d N_1$ and $\Omega_2=a_0^d N_2$ are the bridge and interlayer volumes, respectively. $\Omega=(a_0\xi)^d$ is the volume of the hypercube with the linear size equal to the correlation length ξ. Denoting as φ a potential difference on a distance of the order of ξ we have

$$j_1=\frac{\varphi}{R_1 a_0^{d-1}}, \quad E_1=\frac{\varphi}{a_0 N_1}, \quad j_2=\frac{\varphi}{R_2 a_0^{d-1} N_2}, \quad E_2=\frac{\varphi}{a_0}. \tag{11}$$

Inserting Eq. (11) into Eq. (10) one obtains [8]

$$C_e(\tau>0)=C_1\tau^{-\kappa}+C_2 h^2 \tau^{-w}, \quad p>p_c \tag{12}$$

where $\kappa=d\nu-\alpha_1$, $w=\kappa'+2(t+q)$, and $\kappa'=\nu d-\alpha_2$. The major conclusion that can be read from the above result is that under certain conditions the second term in Eq. (12) can be much greater than the first one. This means that the noise originating from an insulating phase describes the macroscopic noise also above the percolation threshold in spite of the fact that in this region the effective conductivity is described by a metallic component. This phenomenon was suggested by Mantese and Webb [31] and Tremblay, Fourcade, and Breton [7]. Equation (12), which is its quantitative description, was derived by Morozovsky and Snarskii [8]. Other derivations of Eq. (12) have also been proposed [9–12]. They are based on scaling properties that have been assumed for the function $C_e(\tau,h)$. These approaches lead to the main result of Eq. (12), although predictions for the exponents κ and κ' are different. Within the framework of the hierarchical model of the percolation structure considered so far, the critical exponents of $1/f$ noise are

$$\kappa=\nu d-\alpha_1, \quad \kappa'=\nu d-\alpha_2. \tag{13}$$

Scaling analyses predict that κ and κ' are simple combinations of the members of the family of *independent* multifractal exponents and cannot be expressed only in terms of the exponents ν, α_1, and α_2. In spite of this difference the numerical values predicted by both approaches are nearly the same.

TABLE I. Summary of the recent numerical estimates of the transport percolation exponents in 2D and 3D.

	t	q	ν
2D	1.303[a]	1.303[a]	4/3[b]
3D	1.94[c]	0.75[d]	0.88[b]

[a]Reference [32].
[b]Reference [23].
[c]Reference [33].
[d]Reference [34].

In a quite similar way we can obtain the behavior of the noise effective intensity for $p<p_c$ [8],

$$C_e(\tau<0)=C_2|\tau|^{-\kappa'}+C_1 h^2|\tau|^{-w'}, \quad p<p_c \tag{14}$$

where $w'=\kappa+2(t+q)$. Eventually, having C_e both above and below p_c described, we can find C_e also inside the smearing region. The simplest way is to insert $|\tau|=\Delta=h^{-1/(t+q)}$ in either of Eqs. (12) or (14),

$$C_e(|\tau|\sim\Delta)=C_1 h^{-\kappa/(t+q)}+C_2 h^{-\kappa'/(t+q)} \quad \text{for } |\tau|<\Delta. \tag{15}$$

In the next section the above results are verified by means of Monte Carlo simulations of a two-component random resistor network.

III. NUMERICAL SIMULATIONS

In the preceding section the behavior of the $1/f$ noise effective intensity in a TP percolation system has been described. One of the conclusions was that above the percolation threshold either the exponent κ or the exponent $w=2t+2q+\kappa'$ may describe dependence of the effective noise intensity C_e on the concentration p (or τ). Which of the two is observed in a particular system depends on the relation between magnitudes of C_1/C_2 and h. Namely, for

$$\frac{C_1}{C_2} \ll h^{(\kappa-\kappa')/(t+q)} \tag{16}$$

the second term in Eq. (12) prevails over the first one also at the border of the smearing region, i.e., for $|\tau|=\Delta$, and this means that the exponent w describes C_e versus τ above the percolation threshold. Otherwise in C_e versus τ one observes the exponent κ.

It is seen from Eq. (16) that to decide whether the behavior of the $1/f$ noise intensity is described by exponent κ or by exponent w one has to know numerical values of κ and κ'. At first let us note that assuming $\alpha_1=\zeta_R$ and $\alpha_2=\zeta_G$ and putting into Eq. (13) the best numerical estimates of exponents t and q and ν which are gathered in Table I gives $\kappa=1.58$ and $\kappa'=1.01$ in 3D or $\kappa=\kappa'=1.37$ in the 2D case. On the other hand, when $\alpha_1=\alpha_2=1$ is assumed, we get $\kappa=\kappa'=1.64$ in 3D and $\kappa=\kappa'=1.67$ in 2D.

TABLE II. Summary of numerical and renormalization group (RG) estimates of the noise critical exponents κ and κ' in 3D.

	Numerical simulations	RG	Rigorous bounds[a]
κ	1.47,[b] 1.57,[c] 1.58,[d] 1.49,[e] 1.58,[f] 1.46,[g]	1.88,[h] 2.43,[h] 1.83,[i] 2.33,[j] 2.34[k]	$1.53<\kappa<1.6$
κ'	0.55,[l] 0.58,[e] 0.68,[f] 0.74[g]	0.66[m]	$0.38<\kappa'<1.02$

[a]Reference [27].
[b]Reference [35].
[c]Reference [36].
[d]Reference [37], deduced from multifractal exponents.
[e]Reference [9], deduced from multifractal exponents.
[f]Reference [10], deduced from multifractal exponents.
[g]Reference [38], deduced from multifractal exponents.
[h]Reference [40] (cell-to-cell RG).
[i]Reference [40] (two-parameter cell-to-cell RG).
[j]Reference [41] (Migdal-Kadanoff bond moving method).
[k]Reference [42] (cell-to-cell RG).
[l]Reference [13].
[m]Reference [39] (Migdal-Kadanoff bond moving method).

For the 2D case the major conclusion is straightforward. Since $\kappa=\kappa'$ the ratio C_1/C_2 alone decides which of the two terms in Eq. (12) dominates the effective noise intensity C_e. When $C_1 \ll C_2$, noise originating from the insulator and the corresponding exponent w are observed also for $p>p_c$. Since in 3D the choice $\alpha_1=\zeta_R$, $\alpha_2=\zeta_G$ predicts $\kappa>\kappa'$ whereas $\kappa=\kappa'$ for $\alpha_1=\alpha_2=1$, computer simulations are necessary in order to calculate numerical values of κ and κ'. In Table II numerical estimates of κ and κ' in 3D are summarized. It is evident that in 3D $\kappa>\kappa'$ and the condition of Eq. (16) leads to slightly stronger requirements for the exponent w to be observed for $p>p_c$. Using the data of Tables I and II we roughly estimate that $C_2/C_1 \gg h^{-0.3}$ is large enough to make the second term in Eq. (12) greater than the first one at $\tau=\Delta$.

Equations (12), (14), and (15) are valid in the thermodynamic limit, i.e., for $L\to\infty$. In practice, to fulfill this requirement it is sufficient to keep the linear size L of the system much greater than the percolation correlation length ξ. The latter, in the critical region, diverges as $\xi\sim|\tau|^{-\nu}$ and thus it may require a large computational effort to reach the thermodynamic limit in the vicinity of the percolation threshold; lattices of large size have to be simulated. To avoid this, finite size scaling is usually employed. The simplest way to find the size dependence of any quantity for $L \ll \xi$ is to replace $|\tau|$ in thermodynamic equations with $L^{-1/\nu}$. In the case of effective noise intensity C_e we thus obtain

$$C_e(L \ll \xi) \sim C_1 L^{\kappa/\nu} + C_2 h^2 L^{w/\nu} \quad \text{for } p>p_c. \tag{17}$$

Again one of the two terms can dominate the C_e versus L behavior depending on the ratios C_1/C_2 and h. If for the largest size available, i.e., for $L=\xi\sim\Delta^{-\nu}\sim h^{-\nu/(t+q)}$ (note that for the TP system ξ ''saturates'' at $\Delta^{-\nu}$ inside the smearing region), the second term in Eq. (17) is much greater than the first one, the exponent w/ν should be observed on C_e versus L plots. Otherwise the exponent κ/ν should be observed. This leads again to the condition of Eq. (16) and all the above considerations are valid also in the finite size scaling behavior of the effective noise intensity.

Similarly we can analyze the behavior of C_e below the percolation threshold, i.e., for $p<p_c$. If inequality opposite to the one given by Eq. (16) is fulfilled, i.e., if

$$\frac{C_1}{C_2} \gg h^{(\kappa-\kappa')/(t+q)}, \tag{18}$$

then the effective $1/f$ noise intensity C_e originates mainly from the resistance fluctuations in a metallic phase and consequently the exponent w' describes C_e versus τ behavior. Otherwise the noise from an insulator dominates the macroscopic fluctuations and the exponent κ' should be observed [see Eq. (14)]. These conclusions can be easily extended to the finite size scaling behavior of C_e. Numerical studies of this behavior should enable one to distinguish between two C_e versus L dependencies; the one dominated by the exponent w'/ν and the other one dominated by the exponent κ'/ν.

To test theoretical predictions of the preceding section, finite size calculations of the 3D two-phase percolation system have been performed. The preliminary results have already been published [13]. The bonds of a simple cubic lattice were occupied randomly by resistances according to the probability distribution of Eq. (8) with $r_1=1$ and $r_2=1/h=10^7$. The calculations were performed at the percolation threshold $p=p_c=0.2492$ [23] to maximize the percolation correlation length and thus to make the finite size scaling possible. Indeed at p_c we expect $\xi\cong\Delta^{-\nu}=h^{-\nu/(t+q)}\cong h^{-1/3}\sim 10^{7/3}$, which is much greater than the largest lattice size, $L=22$. Once the lattice was generated a unit external voltage was applied to the opposite sides of the lattice. Free boundary conditions were used in the remaining two directions. Next, internal voltages on all the bonds in the lattice were evaluated. The lattice conductance G was calculated as the total power dissipated in the network. The effective noise intensity C_e was calculated with the help of Eq. (3) for various values of the ratio of the component noise intensities. Namely, for the fixed value

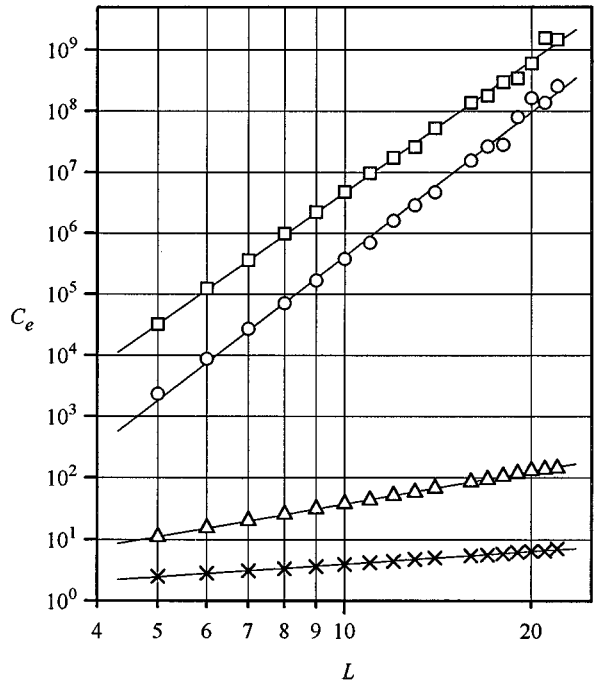


FIG. 2. The effective $1/f$ noise intensity C_e as a function of lattice size L. Simulations were performed for a 3D two-phase percolation system with $r_1=1$ and $r_2=1/h=10^7$ at the percolation threshold $p=p_c=0.2492$. C_e was calculated for various values of the ratio C_1/C_2. Namely, for $C_1=1$ and various values of $C_2=10^{-20}$ (circles) and $C_2=1$ (crosses) for nonpercolating samples and $C_2=10^{20}$ (squares) and $C_2=1$ (triangles) for percolating samples. The slopes of the lines which are the least squares approximations of the data are 7.17 ± 0.2 (squares), 1.75 ± 0.025 (triangles), 7.84 ± 0.02 (circles), and 0.69 ± 0.02 (crosses). C_e is in units of C_1 (triangles), h^2C_2 (squares), C_2 (crosses), and C_1h^2 (circles). L is in units of lattice spacing a_0, C_1, and C_2 are in units of a_0^d/Hz.

$C_1=1$ various values $C_2=10^{-20}$, 1, and 10^{20} were assumed. Thus either of the conditions of Eqs. (16) and (18) could be fulfilled and the ideas of the preceding section were tested numerically. To refer to either above or below threshold behavior we gathered percolating and nonpercolating samples in separate ensembles, in which G and C_e were then averaged. Results are shown in Fig. 2. For percolating samples the slopes of the lines in this figure are 7.17 ± 0.2 and 1.75 ± 0.025 when the condition of Eq. (16) is fulfilled or not, respectively. Thus the exponents w and κ can be estimated as $w=6.31\pm0.25$ and $\kappa=1.54\pm0.025$, in quite good agreement with the predictions of the preceding section, $w=2(t+q)+\kappa'=6.39$ (7.02) and $\kappa=d\nu-\alpha_2=1.58$ (1.64). Similarly, for nonpercolating samples the slopes of the lines are 7.84 ± 0.2 and 0.69 ± 0.02 if the condition given by Eq. (18) is fulfilled or not, respectively. They lead to the estimates of exponents $w'=6.9\pm0.25$ and $\kappa'=0.61\pm0.02$, which are again in quite good agreement with theoretical predictions $w'=2(t+q)+\kappa=6.96$ (7.02) and $\kappa'=d\nu-\alpha_2=1.01$ (1.64).

In conclusion, our numerical simulations confirm the validity of the analysis performed in the preceding section. This strongly supports the usefulness of the hierarchical model of the TP percolation system which has been used in the derivations of theoretical results.

IV. NONLINEAR SUSCEPTIBILITY

Recently a number of papers have appeared which deal with a weakly nonlinear TP random system. In such a system both its components may indicate a weak cubic nonlinearity, which just means that current density depends on electrical field as $\vec{j}_i=\sigma_i\vec{E}_i+\chi_i|E_i|^2\vec{E}_i$, where χ_i is the nonlinear susceptibility of the component i ($i=1,2$). Since the work of Stroud and Hui [43] and Aharony [44] it is known that the effective nonlinear susceptibility χ_e of the whole TP system is related to the fourth moment of the local field distribution, $\chi_e\sim\langle\chi(\vec{r})E(\vec{r})^4\rangle$, and the problem becomes mathematically equivalent to the estimation of effective $1/f$ noise intensity, $\chi_e\sim C_e\sigma_e^2$ for the system with the local noise intensity $C(\vec{r})=\chi(\vec{r})/\sigma^2(\vec{r})$. Thus the critical behavior of χ_e is given immediately from the equations describing the behaviors of the effective noise intensity and the effective conductivity,

$$\chi_e(\tau>0)=C_e(\tau>0)\sigma_e^2(\tau>0)$$
$$=C_e(\tau>0)\sigma_1^2\tau^{2t}=\chi_1\tau^{2t-\kappa}+\chi_2\tau^{-2q-\kappa'}$$
$$\text{for } p>p_c, \tag{19}$$

$$\chi_e(\tau<0)=C_e(\tau<0)\sigma_e^2(\tau<0)$$
$$=C_e(\tau<0)\sigma_2^2\tau^{-2q}$$
$$=\chi_2|\tau|^{-2q-\kappa'}+\chi_1h^4|\tau|^{-w'-2q}$$
$$\text{for } p<p_c, \tag{20}$$

$$\chi_e(|\tau|<\Delta)=C_e(|\tau|<\Delta)\sigma_e^2(|\tau|<\Delta)$$
$$=C_e(|\tau|<\Delta)\sigma_1^{2q/(t+q)}\sigma_2^{2t/(t+q)}$$
$$=\chi_1h^{(2t-\kappa)/(t+q)}+\chi_2h^{-(2q+\kappa')/(t+q)}$$
$$\text{for } |\tau|<\Delta. \tag{21}$$

Thus it is surprising that the authors of Ref. [45] who start just from Eqs. (19)–(21) for the nonlinear susceptibility arrive at the results and conclusions for $1/f$ noise which are different from those given by our Eqs. (12), (14), and (15).

V. 1/f NOISE IN SYSTEMS WITH AN EXPONENTIALLY WIDE SPECTRUM OF RESISTANCES

As has already been mentioned in the Introduction, the distribution which obey resistances forming a random resistor network depends on a structure being modeled. In many physical cases this distribution is very wide even on a logarithmic scale. To be more precise, in this case bonds are occupied in a random way by resistances which obey the form of Eq. (9), where x is a random variable sampled between 0 and 1 according to a smooth distribution $D(x)$ and parameter λ takes on the values much greater than unity, $\lambda\gg1$. It was shown [14–17] that such a network can be solved when treated as the usual percolation problem. In a crude approximation the network effective conductivity, σ_e

is described by the largest resistance which opens the percolating cluster, namely, $\sigma_e \sim \exp(\lambda x_c)$, where x_c is related to the percolation threshold in a classical percolation, i.e.,

$$\int_{x_c}^{1} D(x)dx = p_c$$

is the fraction of bonds required to form a cluster which starts spanning the network. A more detailed treatment leads to the derivation of the preexponential factor in σ_e [17,21,46–50],

$$\sigma_e \sim \lambda^{-y} \exp(\lambda x_c). \tag{22}$$

It was shown by the number of methods [21,47–49] that the exponent y is related to the percolation correlation length exponent ν,

$$y = \nu(d-2). \tag{23}$$

Very recently an approach was proposed [18–20] which also enables one to determine the exponent y. In this approach the system with an exponentially wide spectrum of resistances (EWSR) is treated as a TP percolation system working inside the smearing region [20]. Basing on this, Morozovsky and Snarskii have shown that [19]

$$y = \frac{\alpha_1 - \alpha_2 + 2\nu(d-2)}{2}. \tag{24}$$

Note that in terms of widely accepted values of $\alpha_1 = \alpha_2 = 1$ [21–27], Eq. (24) reduces to Eq. (23). For $\alpha_1 = \zeta_R$ and $\alpha_2 = \zeta_G$ we get, however, $y = (t-q)/2$ [19]. For dimension $d=2$ both the results coincide, giving $y=0$. Using the estimates of Table I we have in the 3D case $\nu(d-2) = \nu \cong 0{,}88$ whereas $(t-q)/2 \cong 0.6$. Numerical simulations of the EWSR system of Tyč and Halperin give $y = 0.6 \pm 0.1$ [48], whereas our recent numerical data give $y = 0.76$ $(+0.09, -0.08)$ [51] and the rejection of any of the above predictions cannot be done.

While, as we have discussed in the previous sections, a number of works deal with $1/f$ noise in TP systems, in EWSR systems investigation has only recently started [19,51]. To our knowledge, until now there has been no unique theory predicting the value of the local noise intensity C_i in the hoppinglike conduction; $1/f$ noise is still the subject of controversy and increasing interest. Nevertheless, there are suggestions that s_i may obey the form of $s_i \sim r_i^{\theta}$. For example, the Hooge phenomenological formula [52], $C \sim 1/n$, where n is a concentration of current carriers, suggests (via $\sigma \sim n$) $s \sim r^1$ and thus $\theta = 1$. Another example is the noise in a single constriction contact where [53] $s \sim r^3$ and thus $\theta = 3$. In fact, in our case the most appropriate values of θ are those predicted by $1/f$ noise generated in tunnel junctions.

To calculate the effective noise intensity of the EWSR system Morozovsky and Snarskii have used a model of the percolation structure in the smearing region [20]. In this model a current flows simultaneously through an ''insulating'' interlayer of resistance R_2 and a ''metallic'' bridge of resistance R_1 which is in series with supplemental resistance r_m, as is seen in Fig. 3. The resistances R_1 and R_2 take

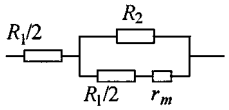


FIG. 3. A model of a two-phase percolation system in the smearing region. Resistances R_1 and R_2 take on fixed values $R_1 = N_1(|\tau| = \Delta) r(x_1)$, $R_2 = N_2(|\tau| = \Delta) r(x_2)$, where $x_1 = x_c + (1 - x_c)\Delta$ and $x_2 = x_c - (1 - x_c)\Delta$. The resistance r_m takes on a random value $r_m = r(x_m)$, where x_m is a random variable which is sampled over the smearing region.

on fixed values which are equal to those achieved at the border of the smearing region, $R_1 = N_1(|\tau| = \Delta) r(x_1)$, $R_2 = N_2(|\tau| = \Delta) r(x_2)$, where $x_1 = x_c + (1 - x_c)\Delta$, $x_2 = x_c - (1 - x_c)\Delta$. Unlike R_1 and R_2 the resistance r_m takes on random values $r_m = r(x_m)$, where x_m is a random variable which, in the case of the EWSR system, is uniformly sampled over the smearing region. Basing on this model it can be shown [19,54] that for $-2 < \theta < 2$ the effective noise intensity obeys the form

$$C_e \sim \lambda^m \exp(-\lambda \theta x_c), \tag{25}$$

where exponent m for $\lambda \to \infty$ and $\theta = 1$ is given by [19]

$$m = y + 2\nu. \tag{26}$$

For $\alpha_1 = \alpha_2 = 1$ we have $y = \nu(d-2)$ and Eq. (26) reads

$$m = d\nu. \tag{26'}$$

As in the case of the exponent y, the estimates (26) and (26′) give the same result $m = 2\nu$ in 2D. In 3D our numerical simulations presented in the next section, although they cannot definitely reject any of the estimates of Eqs. (26) and (26′), fulfill almost exactly the relation $m = 3y$ and this means that the choice $\alpha_1 = \alpha_2 = 1$ seems to be more appropriate to model EWSR systems. This is well understood, since in the EWSR system the ''metallic'' cluster is formed from bonds with $r_i \leqslant r_c$. r_c is obviously the resistance of the bond which opens the percolating cluster, i.e., it is SCB. Other metallic bonds which are mostly in blobs of the metallic cluster have much smaller resistances (due to a wide resistance distribution) and thus the influence of blobs on transport properties is not so large as in TP percolation; the flow of a current is governed by the SCB's rather than by the whole backbone of the percolation cluster.

Very recently an interesting feature of the exponent m describing the preexponential factor in Eq. (25) has been observed [54]. It has turned out that m is θ independent although the noise intensity itself is strongly (exponentially) θ dependent. Thus the exponent m emerges as a DOUBLE UNIVERSAL. Apart from the usual percolation universality, i.e., independence of the lattice geometry, it is also independent of the microscopic noise generating mechanism. This conjecture is confirmed also by numerical simulations presented in the next section.

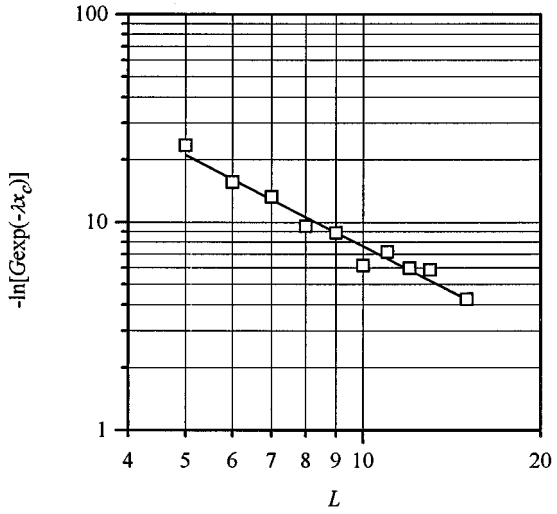


FIG. 4. Finite size scaling of the conductance G (in units of $g_0=1/r_0$) of 3D lattice of the size L with an exponentially wide spectrum of resistances. The points are the results of numerical simulations. G is the average in the ensemble of several hundred realizations for $L=15$ up to several thousand realizations for $L=5$. On the vertical axis the value $x_c=1-p_c=1-0.2492$ was used in the rescaling of conductance. Simulations were performed for $\lambda=140$, which causes the system to be not in a homogeneous region, since $\xi\cong\lambda^{\nu}=85\gg L$ (both L and ξ are in units of lattice spacing a_0). The slope of the line which is the least squares fit to the data is -1.46.

VI. NUMERICAL SIMULATIONS OF SYSTEMS WITH AN EXPONENTIALLY WIDE SPECTRUM OF RESISTANCES

To test results from the preceding section we have performed computer simulations of a 3D EWSR system. In each computational step a simple cubic lattice of linear size L, in which bonds were occupied randomly in the way described in Eq. (9), was generated. A uniform distribution $D(x)=1$ was assumed for simplicity. Once the lattice was generated, resistances r_i of all its bonds were stored in a band matrix of network equations and a unit dc external voltage was applied to the opposite walls of the lattice. Free boundary conditions were applied in the remaining two directions. Then voltages U_i on all bonds of the lattice were computed. Once the voltages were evaluated the network conductance G was calculated. Then the band matrix was refilled with the local RPSD's, s_i, calculated according to the form $s_i=r_i^{\theta}$. Next power spectral density of the network conductance fluctuations, $S_G=\{\delta G\,\delta G\}=SG^2$, was calculated with the help of Eq. (3).

Let us now consider whether, as in the case of the TP system, an employment of finite size scaling in the calculation of exponent m is possible. At first let us note that it is relatively easy to follow the ideas of Kurkijärvi [47] to get the result $m=d\nu$ [55]. In this approach the only finite size effect which is considered is the dependence of the percolation threshold on the system size L,

$$p_{cL}=p_c+\text{const}\;L^{-1/\nu}, \tag{27}$$

while another effect of the increasing length of a percolation path is not taken into account. However, the latter results in

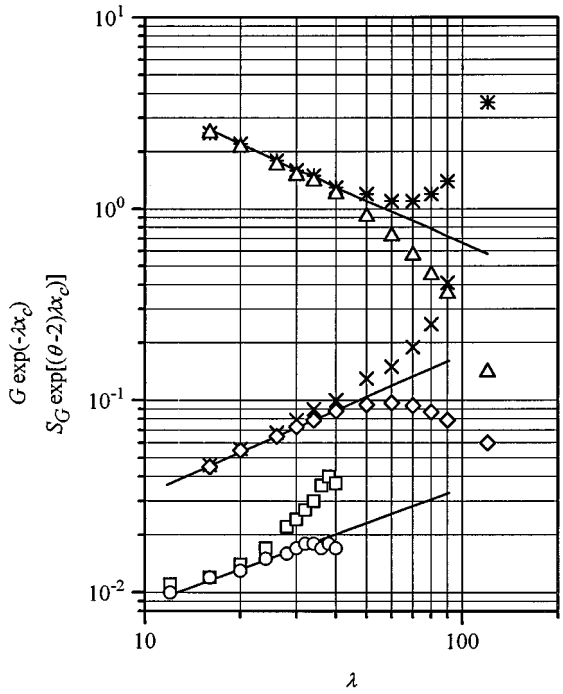


FIG. 5. Results of numerical studies of conductance G and power spectral density of conductance fluctuations S_G in random resistor network with an exponentially wide spectrum of resistances. Points are results of Monte Carlo simulations of a simple cubic lattice of size $L=15$. Lines are drawn according to Eqs. (22) and (25). Their slopes, i.e., exponents $-y$ and $m-2y$, are calculated from the least squares analysis of the data in the range where both arithmetic (extensions $\_a$) and harmonic averages (extensions $\_h$) coincide, i.e., for $16\leqslant\lambda\leqslant30$. Points refer to G_a (stars), G_h (triangles), S_G_a for $\theta=1$ (crosses), S_G_h for $\theta=1$ (diamonds), S_G_a for $\theta=0$ (squares), S_G_h for $\theta=0$ (circles). In rescaling of G and S_G the value of $1-0.2492$ was used as the best known estimate of the percolation threshold $x_c=1-p_c$. G is in units of $g_0=1/r_0$, S_G is in units of g_0^2/Hz, L is in units of a_0

the increasing of resistance of the percolation cluster. Within the framework of the hierarchical model of Fig. 1 the resistance R_1 of a metallic bridge in a system of size $L\ll\xi$ scales as

$$R_1=N_1r_1\sim L^{\alpha_1/\nu}\lambda^{-1}r(x_{cL}), \tag{28}$$

where in place of r_1 we set $\langle r\rangle_1$, the average resistance of bonds which form the percolation cluster,

$$\langle r\rangle_1=\int_{x_{cL}}^{1}r(x)D(x)dx\cong\frac{1}{\lambda}\,r(x_{cL}). \tag{29}$$

Thus preexponential and exponential L dependencies are involved in the finite size scaling behavior of the overall conductance of the EWSR system,

$$G\sim\frac{1}{R_1}\sim L^{-\alpha_1/\nu}\lambda\;\exp(\lambda x_c)\exp(-\text{const}\;\lambda L^{-1/\nu}). \tag{30}$$

In view of this an exponential dependence with exponent $-1/\nu$ is expected to dominate G versus L rather than a power-law dependence with exponent $-\alpha_1/\nu$. Indeed, the least squares fit of the data to the straight line given by $-\ln[G\exp(-\lambda x_c)]$ versus L in log-log coordinates shown in Fig. 4 leads to the acceptable estimate of the exponent $\nu=0.68\pm0.1$. A similar finite size scaling behavior exhibits also the effective noise intensity C_e and thus we may conclude that in our approach finite size scaling cannot be used to evaluate exponents y or m. They should rather be calculated directly from σ_e versus λ or C_e versus λ relations in the thermodynamic limit where Eqs. (22) and (25) are valid. Therefore our simulations were performed for various values of the parameter λ in the range from 10 to 80 and for the lattice size $L=15$. For each value of λ, up to several hundred realizations of the network were generated and their conductances were averaged in two different ways, namely, arithmetic and harmonic averages were calculated. Then the data were arranged into plots shown in Fig. 5. Basing on this figure, we can check whether the data fall into the thermodynamic limit by looking at the differences between arithmetic and harmonic averages. When those two coincide it means that all network realizations give practically one value of G and another one for S_G. The system is then in the homogeneous region and we may use Eqs. (22) and (25) to approximate the numerical data. Thus when we plot products $G\exp(-\lambda x_c)$ and $S_G\exp[\lambda(\theta-2)x_c]$ (note that for L fixed, $S_G=SG^2\sim C_e\sigma_e^2\sim\lambda^{m-2y}\exp[-\lambda(\theta-2)x_c]$ as a function of λ in log-log scale, straight lines with the slopes $-y$ and $m-2y$ should be observed. This is really the case for $\lambda\leqslant30$ in Fig. 5. For $\lambda=30$ where the differences start becoming significant we have $\xi\cong\lambda^{\nu}\cong20$, which is larger than our lattice size L and the system is no longer in the thermodynamic limit. The least squares analysis of the data in the range $16\leqslant\lambda\leqslant30$ gives $y=0.76\pm0.04$ and $m-2y=0.78\pm0.09$ for $\theta=0$ and $m-2y=0.73\pm0.3$ for $\theta=1$. In the calculations the value of $x_c=1-p_c\cong1-0.2492$ was used as the best known estimate of the percolation threshold in the simple cubic lattice [23].

As a final result of simulations we get approximately $m=2.25\pm0.38$ for $\theta=0$ and $m=2.3\pm0.17$ for $\theta=1$. These values should be compared with the theoretical analysis of the preceding section which for $\alpha_1=\alpha_2=1$ predicts $m=3\nu\cong2.64$ whereas for $\alpha_1=\zeta_R$ and $\alpha_2=\zeta_G$ $m=y+2\nu\cong2.36$. Independently of the values of α_1 and α_2 theory predicts double universality of the exponent m. Since we have obtained $m(\theta=1)\cong m(\theta=0)$ one can conclude that this double universality has been confirmed by our numerical simulations.

VII. CONCLUSIONS

The behavior of the $1/f$ noise effective intensity in a two-phase percolation system and percolationlike system with an exponentially wide distribution of bond resistances has been reviewed. Monte Carlo simulations have been performed. For a two-phase system numerical values of noise critical exponents $\kappa=1.54\pm0.025$, $\kappa'=0.61\pm0.02$, $w=6.31\pm0.25$, and $w'=6.9\pm0.25$ have been found in agreement with the analysis performed with the help of a hierarchical model of a two-phase percolation system. In a system with an exponentially wide spectrum of resistances it has been found that for a microscopic noise generating mechanism which obeys the form of $\{\delta r^2\}\sim r^{2+\theta}$ the effective noise intensity is given by $C_e\sim\lambda^m\exp(-\lambda\theta x_c)$ where $1-x_c$ is the percolation threshold. The exponent m is ''double universal,'' i.e., it is independent of lattice geometry and of the microscopic noise generating mechanism. Numerical simulations performed for $\theta=1$ and 0 give $m\cong2.3$ and confirm double universality of the exponent m. Connections of $1/f$ noise effective intensity with the effective nonlinear susceptibility in the two-phase weakly nonlinear percolation system have also been established.

ACKNOWLEDGMENTS

The work was supported in part by the Commission for Scientific Research of Poland through Grant No. 8T11B 038 09 and in part by the Russian Foundation for Fundamental Research through Grant No. 95-02-04432-a. A.A.S. is also indebted to Rzeszów University of Technology for hospitality, where part of this manuscript was prepared. The authors are grateful for useful discussions with A. W. Stadler.

[1] For a review of $1/f$ noise phenomenon see, e.g., S. M. Kogan, Usp. Fiz. Nauk **145**, 285 (1985) [Sov. Phys. Usp. **28**, 170 (1985)]; M. B. Weissman, Rev. Mod. Phys. **60**, 537 (1987).

[2] H. Butterweck, Philips Res. Rep. **30**, 316 (1975).

[3] R. Rammal, C. Tannous, P. Breton, and A.-M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985); Phys. Rev. A **31**, 2662 (1985).

[4] A. L. Efros and B. Shklovskii, Phys. Status Solidi B **76**, 475 (1976).

[5] J. P. Straley, J. Phys. C **9**, 783 (1976); Phys. Rev. B **15**, 5733 (1977); J. Phys. C **12**, 3711 (1979).

[6] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

[7] A.-M. S. Tremblay, B. Fourcade, and P. Breton, Physica A **157**, 89 (1989).

[8] A. E. Morozovsky and A. A. Snarskii, Zh. Eksp. Teor. Fiz. **95**, 1844 (1989) [Sov. Phys. JETP **68**, 1066 (1989)]; Pis'ma Zh. Eksp. Teor. Fiz. **52**, 871 (1990) [JETP Lett. **52**, 244 (1990)]; Zh. Eksp. Teor. Fiz. **102**, 683 (1992), [Sov. Phys. JETP **75**, 366 (1992)].

[9] R. R. Tremblay, G. Albinet, and A.-M. S. Tremblay, Phys. Rev. B **43**, 11 546 (1991).

[10] R. R. Tremblay, G. Albinet, and A.-M. S. Tremblay, Phys. Rev. B **43**, 775 (1992).

[11] A.-M. S. Tremblay, R. R. Tremblay, G. Albinet, and B. Fourcade, Physica A **183**, 398 (1992).

[12] A. Kolek, Phys. Rev. B **45**, 205 (1992).

[13] A. Kolek, Int. J. Electron. **73**, 1095 (1992).

[14] B. I. Shklovskii and A. L. Efros, *Electronic Properties of Doped Semiconductors* (Springer, Berlin, 1984).

[15] V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).

[16] M. Pollack, J. Non-Cryst. Solids **11**, 1 (1972).
[17] B. I. Shklovskii and A. L. Efros, Zh. Eksp. Teor. Fiz. **60**, 867 (1971) [Sov. Phys. JETP **33**, 468 (1971)].
[18] A. E. Morozovsky and A. A. Snarskii, Pis'ma Zh. Eksp. Teor. Fiz. **56**, 272 (1992) [JETP Lett. **56**, 268 (1992)].
[19] A. E. Morozovsky and A. A. Snarskii, Zh. Eksp. Teor. Fiz. **104**, 4059 (1993) [JETP **77**, 959 (1993)].
[20] A. E. Morozovsky and A. A. Snarskii, Int. J. Electron. **78**, 135 (1995); note that the critical exponent in Eq. (5) of this reference should read 2ν instead of 2υ.
[21] A. Skal and B. I. Shklovskii, Fiz. Tekh. Poluprovodn. **8**, 1586 (1974) [Sov. Phys. Semicond. **8**, 1029 (1975)].
[22] P. G. de Gennes, J. Phys. (Paris) **37**, L21 (1976).
[23] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).
[24] A. Coniglio, Phys. Rev. Lett. **46**, 250 (1981); in *Disordered Systems and Localization*, edited by C. Castellani, C. Di Castro, and L. Petiti, Lecture Notes in Physics Vol. 149 (Springer, Berlin, 1982), p. 51.
[25] V. E. Dubrov, M. E. Levinstein, and M. S. Shur, Zh. Eksp. Teor. Fiz. **70**, 2014 (1976) [Sov. Phys. JETP **43**, 1050 (1976)].
[26] A. Coniglio and H. E. Stanley, Phys. Rev. Lett. **52**, 1069 (1984).
[27] D. C. Wright, D. J. Bergman, and Y. Kantor, Phys. Rev. B **33**, 396 (1986).
[28] A. A. Snarskii, Zh. Eksp. Teor. Fiz. **91**, 1405 (1986) [Sov. Phys. JETP **64**, 828 (1986)].
[29] See, e.g., B. J. Balagurov, Fiz. Tekh. Poluprovodn. **20**, 1276 (1986) [Sov. Phys. Semicond. **20**, 1276 (1986)].
[30] R. Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem* (McGraw-Hill, New York, 1967).
[31] J. V. Mantese and W. W. Webb, Phys. Rev. Lett. **55**, 2212 (1985).
[32] D. J. Frank and C. J. Lobb, Phys. Rev. B **37**, 302 (1988).
[33] B. Derrida, D. Stauffer, H. J. Herrmann, and J. Vannimenus, J. Phys. (Paris) Lett. **44**, L701 (1983).
[34] H. J. Herrmann, B. Derrida, and J. Vannimenus, Phys. Rev. B **30**, 4080 (1984).
[35] A. Kolek and A. Kusy, J. Phys. C **21**, L573 (1988).
[36] I. Balberg, Philos. Mag. B **57**, 991 (1987).
[37] E. Duering and D. J. Bergman, J. Stat. Phys. **60**, 363 (1990).
[38] A. Kolek (unpublished).
[39] P. M. Hui and D. Stroud, Phys. Rev. B **34**, 8101 (1986).
[40] A. Kolek and A. Kusy, in *Proceedings of the 10th International Conference on Noise in Physical Systems 1989*, edited by A. Ambrozy (Budapest Akademia Kiado, Budapest, 1984), p. 343.
[41] R. Rammal, J. Phys. (Paris) Lett. **46**, 129 (1985).
[42] J. M. Luck, J. Phys. A **18**, 2061 (1985).
[43] D. Stroud and P. M. Hui, Phys. Rev. B **37**, 8719 (1988).
[44] A. Aharony, Phys. Rev. Lett. **58**, 2726 (1987).
[45] O. Levy and D. J. Bergman, Phys. Rev. B **50**, 3652 (1994).
[46] V. Ambegaokar, S. Cochran, and J. Kurkijärvi, Phys. Rev. B **8**, 3682 (1973).
[47] J. Kurkijärvi, Phys. Rev. B **9**, 770 (1973).
[48] S. Tyč and B. I. Halperin, Phys. Rev. B **39**, 877 (1989).
[49] P. LeDoussal, Phys. Rev. B **39**, 881 (1989).
[50] E. Charlaix, E. Guyon, and S. Roux, in *Transport in Porous Media 2* (Reidel, Dordrecht, 1987), p. 31.
[51] A. Kolek, A. A. Snarskii, and A. E. Morozovsky, Zh. Eksp. Teor. Fiz. **108**, 894 (1995) [JETP **81**, 490 (1995)].
[52] F. N. Hooge, Phys. Lett. A **29**, 139 (1969).
[53] L. K. Vandamme, Electrocomponent Sci. Technol. **4**, 171 (1977).
[54] A. A. Snarskii and A. Kolek (unpublished).
[55] A. Kolek, in *Proceedings of the XVII-th National Conference on Circuit Theory and Electronic Networks*, edited by W. Wolski (Oficyna Wydawnicza Politechniki Wroclawskiej, Wroclaw, 1994), p. 25.